Part 2: Question 1 (20 points)

Consider an economy with $I = 2$ consumers and $L = 2$ goods. The utility functions for both consumers are:

$$u_1(\boldsymbol{x}_1) = -\frac{4}{x_{11}} + x_{21}$$
$$u_2(\boldsymbol{x}_2) = -\frac{16}{x_{12}} + x_{22}$$

where $x_{\ell i}$ represents consumer i's consumption of the ℓth good. The price of good 2 is normalized to 1 and the price of good 1 is p. Each consumer has no endowment of good 1 and an initial endowment of good 2 equal to ω_{2i} where $\omega_{2i} > \sqrt{8}$ for $i = 1, 2$. Each consumer's consumption set is $X_i = \mathbb{R}^2_+$.

There are 3 competitive firms that can produce good 1 from good 2 according to the following production technology:

$$\mathcal{Y}_j = \left\{\boldsymbol{y}_j \in \mathbb{R}^2 : y_{1j} \leq (-3y_{2j})^{\frac{1}{3}}, y_{2j} \leq 0\right\}$$

In words: z units of good 2 can produce up to $(3z)^{\frac{1}{3}}$ units of good 1. Each consumer has an equal share in the profits of each firm.

(i) [**4 points**] Find the aggregate supply function.

(ii) [**4 points**] Find the profit function of each firm.

(iii) [**4 points**] Find the aggregate excess demand function.

(iv) [**4 points**] Show that the aggregate excess demand function **does not** have the gross-substitute property.

(v) [**4 points**] Find the equilibrium price of good 1.

Solution:

(i) Firm j's problem is:

$$\max_{\boldsymbol{y}_j \in \mathcal{Y}_j} py_{1j} + y_{2j}$$

If $p > 0$, then the firm sets $y_{1j} = (-3y_{2j})^{\frac{1}{3}}$. Solving for y_{2j} yields: $y_{2j} = -\frac{1}{3}y_{1j}^3$. Substituting this in:

$$\max_{y_{1j} \geq 0} py_{1j} - \frac{1}{3}y_{1j}^3$$

The first-order condition is $p - y_{1j}^2 = 0$. Firm j's optimal supply of good 1 is then $y_{1j}(p) = \sqrt{p}$. Firm j's optimal supply of good 2 is then: $y_{2j}(p) = -\frac{1}{3}y_{1j}^3 = -\frac{1}{3}p^{\frac{3}{2}}$. The aggregate supply function is then:

$$\boldsymbol{y}(p) = \sum_{j=1}^{3} \begin{pmatrix} \sqrt{p} \\ -\frac{1}{3}p^{\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 3\sqrt{p} \\ -p^{\frac{3}{2}} \end{pmatrix}$$

(ii) Using the supply function in part (i), the profit function for firm j is:

$$\begin{aligned} \pi_j(p) &= p\sqrt{p} - \frac{1}{3}p^{\frac{3}{2}} \\ &= p^{\frac{3}{2}} - \frac{1}{3}p^{\frac{3}{2}} \\ &= \frac{2}{3}p^{\frac{3}{2}} \end{aligned}$$

Each consumer therefore receives $\frac{1}{2} \times 3 \times \frac{2}{3}p^{\frac{3}{2}} = p^{\frac{3}{2}}$ from these profits.

(iii) Let $b_1 = 4$ and $b_2 = 16$. Consumer i's problem is:

$$\max_{\boldsymbol{x}_i} -\frac{b_i}{x_{1i}} + x_{2i} \text{ subject to } px_{1i} + x_{2i} \leq \frac{1}{2}\sum_{j=1}^{3} \pi_j(p) + \omega_{2i} = p^{\frac{3}{2}} + \omega_{2i}$$

Since utility is increasing in both goods, the budget constraint will bind. Substituting the constraint solved for x_{2i} into the objective:

$$\max_{x_{1i} \geq 0} -\frac{b_i}{x_{1i}} + p^{\frac{3}{2}} + \omega_{2i} - px_{1i}$$

The first-order condition is $\frac{b_i}{x_{1i}^2} - p \leq 0$, with equality if $x_{1i} > 0$. $x_1 = 0$ would violate this inequality so we get an interior solution of $x_{1i}(p) = \sqrt{\frac{b_i}{p}}$. The demand for good 2 is then $x_{2i}(p) = p^{\frac{3}{2}} + \omega_{2i} - \sqrt{b_i p}$. Aggregate excess demand is then:

$$\boldsymbol{z}(p) = \boldsymbol{z}_1(p) + \boldsymbol{z}_2(p) = \begin{pmatrix} \sqrt{\frac{4}{p}} \\ p^{\frac{3}{2}} - \sqrt{4p} \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{16}{p}} \\ p^{\frac{3}{2}} - \sqrt{16p} \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{p}} \\ 2p^{\frac{3}{2}} - 6\sqrt{p} \end{pmatrix}$$

(iv) The gross-substitute property here would be that $\frac{\partial z_2(p)}{\partial p} > 0$. Taking this derivative:

$$\frac{3}{2}2p^{\frac{3}{2}-1} - \frac{1}{2}6p^{\frac{1}{2}-1} = 3p^{\frac{1}{2}} - 3p^{-\frac{1}{2}} = 3\left(\sqrt{p} - \frac{1}{\sqrt{p}}\right) =$$

If $p > 1$, then $\frac{\partial z_2(p)}{\partial p} > 0$. But if If $p \in (0,1)$, then $\frac{\partial z_2(p)}{\partial p} < 0$. Therefore it does not posses the gross-substitute property.

(v) For an equilibrium, we need to find the price p that solves $\boldsymbol{z}(p) = \boldsymbol{y}(\boldsymbol{p})$. We only need one equation to do this. We do it with good 1. We need to find the p that solves $3\sqrt{p} = \frac{6}{\sqrt{p}}$, which is $p = 2$. We can confirm that $p = 2$ satisfies equilibrium in the 2nd good. Supply is $-p^{\frac{3}{2}} = -2^{\frac{3}{2}}$ and demand is:

$$2p^{\frac{3}{2}} - 6\sqrt{p} = \underbrace{2 \times 2^{\frac{3}{2}}}_{=2\times 2\times\sqrt{2}} - 6 \times \sqrt{2} = (4-6)\sqrt{2} = -2\sqrt{2}$$

So the market for good 2 also clears.

Part 2: Question 2 (20 points)

Consider a pure exchange economy with L goods and I consumers with preferences $\succeq_i$ over $X_i = \mathbb{R}_+^L$ such that the aggregate excess demand function $\boldsymbol{z}(\boldsymbol{p})$ is continuous over all $\boldsymbol{p} \in \mathbb{R}_+^L$ and satisfies:

- $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{z}(\alpha \boldsymbol{p})$ for all $\alpha > 0$ (homogeneity of degree zero).
- $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$ for all $\boldsymbol{p} \in \mathbb{R}_+^L$.

Notice that Walras' law is not satisfied for all price vectors. We do not have local nonsatiation.

The distribution of the endowment vector satisfies $\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i. There is a single firm whose production set is:

$$Y_1 = \left\{ \boldsymbol{y}_1 \in \mathbb{R}^L : \boldsymbol{y_1} \leq \mathbf{0} \right\}$$

Each consumer i is entitled to a share $\theta_{i1} = \frac{1}{I}$ of the firm's profits.

Define the following function $\boldsymbol{f} : \Delta \to \Delta$:

$$\boldsymbol{f}(\boldsymbol{p}) = \{f_\ell(\boldsymbol{p})\}_{\ell=1}^L = \left\{ \frac{p_\ell + \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{k=1}^L \max\{0, z_k(\boldsymbol{p})\}} \right\}_{\ell=1}^L$$

where:

$$\Delta = \left\{ \boldsymbol{p} \in \mathbb{R}_+^L : \sum_{\ell=1}^L p_\ell = 1 \right\}$$

(i) [**2 points**] Show that Δ is non-empty.

(ii) [**2 points**] Show that Δ is convex.

(iii) [**2 points**] Show that Δ is compact.

(iv) [**2 points**] Show that if $\boldsymbol{p} \in \Delta$, then $f(\boldsymbol{p}) \in \Delta$.

(v) [**2 points**] Provide some argumentation that $\boldsymbol{f} : \Delta \to \Delta$ is continuous. You do not need to give a full rigorous proof.

(vi) [**5 points**] Prove that there exists a price vector $\boldsymbol{p}^\star \geq \mathbf{0}$, $\boldsymbol{p}^\star \neq \mathbf{0}$ that satisfies $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$.

(vii) [**5 points**] If there is a price vector $\boldsymbol{p}^\star \geq \mathbf{0}$, $\boldsymbol{p}^\star \neq \mathbf{0}$ that satisfies $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$, is it necessarily a Walrasian equilibrium price vector?

Solution:

(i) Take $\boldsymbol{p} = \frac{1}{L}$ for all ℓ. This satisfies $\boldsymbol{p} \in \mathbb{R}^L_+$ and $\sum_{\ell=1}^{L} p_\ell = 1$. Therefore $\boldsymbol{p} \in \Delta$. Therefore Δ is non-empty.

(ii) Take $\boldsymbol{p}' \in \Delta$ and $\boldsymbol{p}'' \in \Delta$. We need to show that for any $\alpha \in [0,1]$ that $\boldsymbol{p}^\alpha = \alpha \boldsymbol{p}' + (1-\alpha)\boldsymbol{p}'' \in \Delta$. Each element of $\boldsymbol{p}^\alpha$ will be nonnegative by the nonnegativity of $\boldsymbol{p}'$, $\boldsymbol{p}''$ and $\alpha \in [0,1]$. Furthermore, $\sum_{\ell=1}^{L} p_\ell^\alpha = 1$ because:

$$\sum_{\ell=1}^{L} p_\ell^\alpha = \sum_{\ell=1}^{L} \left[\alpha p_\ell' + (1-\alpha) p_\ell''\right] = \alpha \sum_{\ell=1}^{L} p_\ell' + (1-\alpha) \sum_{\ell=1}^{L} p_\ell'' = \alpha + (1-\alpha) = 1$$

Therefore Δ is convex.

(iii) Δ is bounded because each p_ℓ must be nonnegative and sum to 1 (so no element of $\boldsymbol{p}$ can be unbounded). It is also closed. Take an arbitary sequence $\boldsymbol{p}^k$ converging to $\boldsymbol{p}$ where $\boldsymbol{p}^k \in \Delta$ for all k. We know that $p_\ell^k \geq 0\ \forall \ell$ and $\sum_{\ell=1}^{L} p_\ell^k = 1$ for all steps k in the sequence. Limits preserve weak inequalities and equalities so $\boldsymbol{p} \in \Delta$ as well. So Δ contains all of its limit points and is thus closed. Δ is closed and bounded so it is compact.

(iv) If $\boldsymbol{p} \in \Delta$, then:

$$\sum_{\ell=1}^{L} f_\ell(\boldsymbol{p}) = \sum_{\ell=1}^{L} \frac{p_\ell + \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{k=1}^{K} \max\{0, z_k(\boldsymbol{p})\}} = \frac{\overbrace{\sum_{\ell=1}^{L} p_\ell}^{=1} + \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{k=1}^{K} \max\{0, z_k(\boldsymbol{p})\}} = 1$$

And because $p_\ell \geq 0$ and $\max\{0, z_\ell(\boldsymbol{p})\} \geq 0$, and the denominator is strictly positive, we also have $f_\ell(\boldsymbol{p}) \geq 0$ for all ℓ. So $f(\boldsymbol{p}) \in \Delta$ as well.

(v) By the continuity of each of the $z_\ell(\boldsymbol{p})$ and because the sum of continuous functions is continuous, the numerator and denominator of the fraction is continuous. The denominator is also bounded away from zero because it takes a minimum value at 1. Because a fraction of two continuous functions is continuous (provided the denominator is bounded away from zero), $f(\boldsymbol{p})$ is continuous. For a formal proof, see the class notes on Canvas.

(vi) From (iv), the function $\boldsymbol{f}(\boldsymbol{p})$ is continuous and maps vectors from a compact, convex and nonempty set to itself. Therefore by Brouwer it has a fixed point. Let $\boldsymbol{p}^\star$ be the fixed point. We just need to show that this fixed point price vector satisfies $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0}$.

For each good we know that:

$$p_\ell^\star = \frac{p_\ell^\star + \max\{0, z_\ell(\boldsymbol{p}^\star)\}}{1 + \sum_{k=1}^K \max\{0, z_k(\boldsymbol{p}^\star)\}}$$

Multiplying across the denominator on the right hand side and canceling the $p_\ell^\star$ terms (just like in class):

$$p_\ell^\star \left(\sum_{k=1}^K \max\{0, z_k(\boldsymbol{p}^\star)\} \right) = \max\{0, z_\ell(\boldsymbol{p}^\star)\}$$

Multiplying both sides by $z_\ell(\boldsymbol{p}^\star)$ and summing over ℓ (just like in class):

$$\sum_{\ell=1}^L z_\ell(\boldsymbol{p}^\star)\, p_\ell^\star \left(\sum_{k=1}^K \max\{0, z_k(\boldsymbol{p}^\star)\} \right) = \sum_{\ell=1}^L z_\ell(\boldsymbol{p}^\star) \max\{0, z_\ell(\boldsymbol{p}^\star)\}$$

Looking at the left hand side:

$$\underbrace{\underbrace{\left(\sum_{k=1}^K \max\{0, z_k(\boldsymbol{p}^\star)\} \right)}_{\geq 0} \underbrace{\sum_{\ell=1}^L z_\ell(\boldsymbol{p}^\star)\, p_\ell^\star}_{\leq 0 \text{ from assumptions}}}_{\leq 0} = \sum_{\ell=1}^L z_\ell(\boldsymbol{p}^\star) \max\{0, z_\ell(\boldsymbol{p}^\star)\}$$

So we know that:

$$\sum_{\ell=1}^L z_\ell(\boldsymbol{p}^\star) \max\{0, z_\ell(\boldsymbol{p}^\star)\} \leq 0$$

Suppose there were any $z_\ell(\boldsymbol{p}^\star) > 0$. Then those elements of the sum over ℓ would be positive (squaring a positive number is positive). But no part of the sum can be negative: if $z_\ell(\boldsymbol{p}^\star)$ is negative, then that element of the sum is zero. So there can't be any positive $z_\ell(\boldsymbol{p}^\star)$ because otherwise the sum would never be nonpositive. Therefore $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0}$.

(vii) Such a price vector is not necessarily a Walrasian equilibrium price vector. Walras' law was one of the necessary conditions for the existence proof we did in class, which is missing here. Walras' law says $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$ for all $\boldsymbol{p} \in \mathbb{R}_+^L$. Here we only have $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$ for all $\boldsymbol{p} \in \mathbb{R}_+^L$.

Because $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0}$, we know that $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0} \in Y_1$. Therefore producing $\boldsymbol{z}(\boldsymbol{p}^\star)$ is feasible for the firm. But because we don't have Walras' law, it is possible for

$z_\ell(\boldsymbol{p}^\star) < 0$ while $p^\star_\ell > 0$ for some ℓ under the above assumptions. Therefore it is possible that $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) < 0$, which means the firm makes negative profits. The firm can always do nothing which gives zero profits: $\boldsymbol{y}_1 = \mathbf{0} \in Y_1$ and $\boldsymbol{p} \cdot \boldsymbol{y}_1 = \boldsymbol{p} \cdot \mathbf{0} = 0$ for all $\boldsymbol{p}$. Therefore producing $\boldsymbol{z}(\boldsymbol{p})$ may not be profit-maximizing for the firm given its technology, meaning it is not an equilibrium.